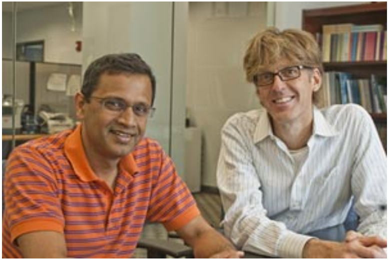
Between Linearizability and Quiescent Consistency



Radha Jagadeesan James Riely

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Chicago, USA

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Linearizability (Herlihy/Wing 1990)

- “Each method call should appear to take effect instantaneously at some moment between its invocation and response.”
  (Herlihy/Shavit 2008)
- Ie, for every invocation, exists a *linearization point* such that
  - linearization point is between call and return
  - real-time order corresponds to some sequential execution

- Compositional (Herlihy/Wing 1990)
  Composition of the histories of two non-interfering linearizable objects is linearizable
- Intrinsically inefficient (Dwork/Herlihy/Waarts 1997)
  Trade-off between high contention and using many variables
  Data Structures in the Multicore Age (Shavit 2011, CACM)

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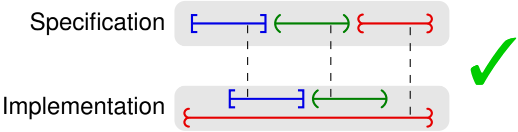

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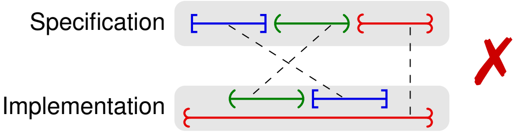

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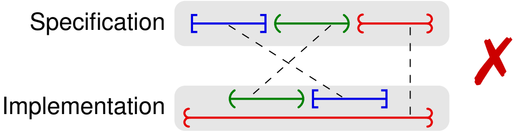

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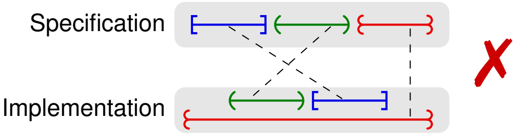

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- "Method calls separated *by a period of quiescence* should appear to take effect in their real-time order."
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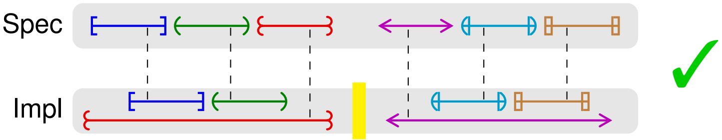

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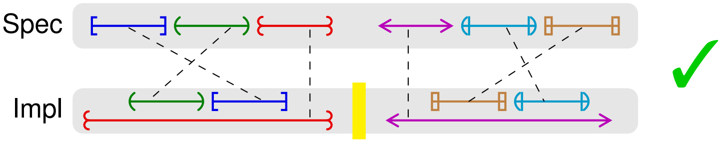

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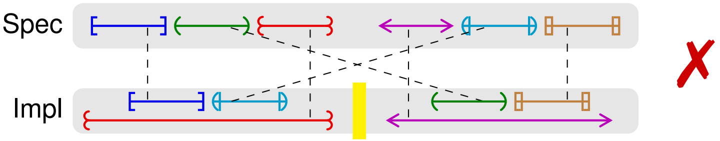

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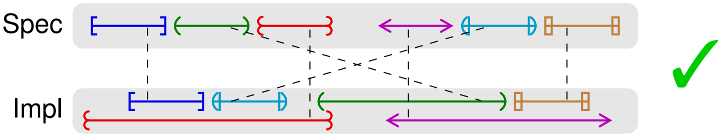

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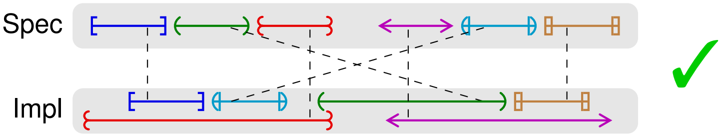

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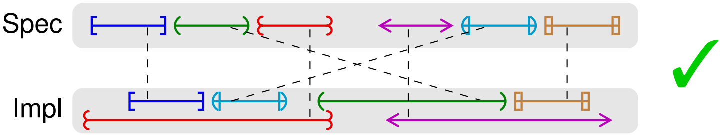

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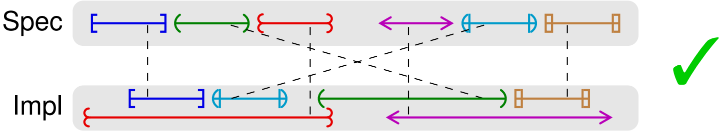

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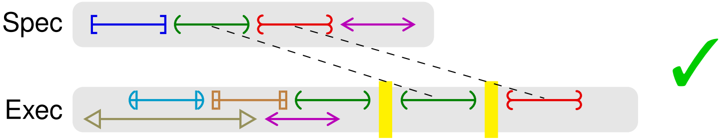

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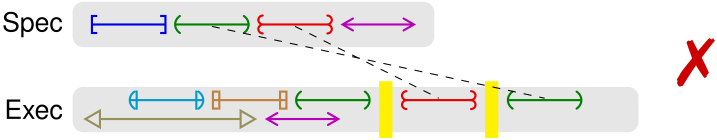

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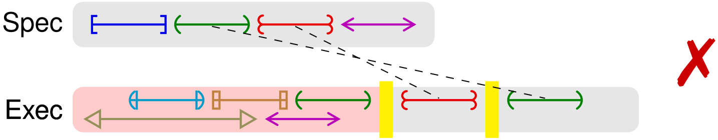

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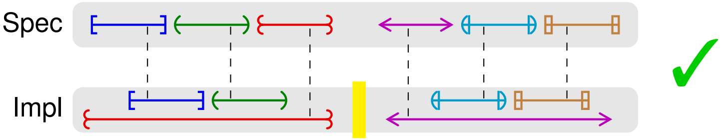

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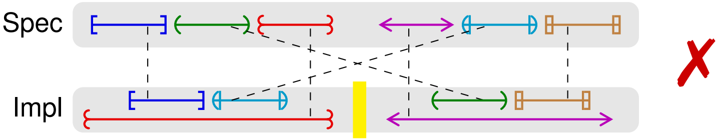

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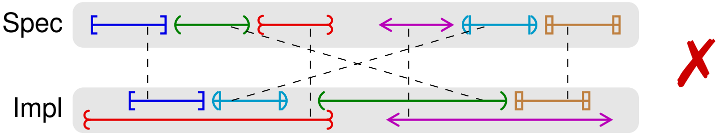

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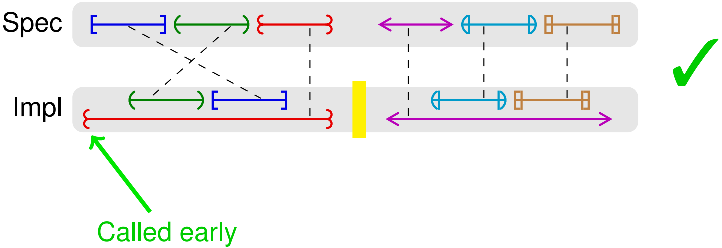

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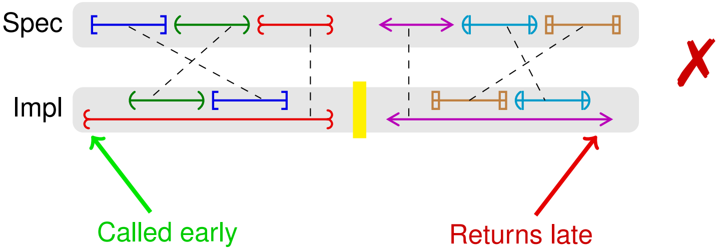

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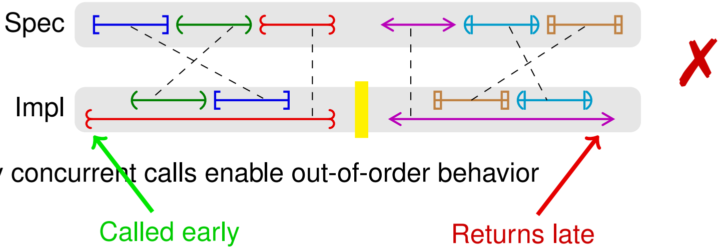

- Early concurrent calls enable out-of-order behavior

Definitions

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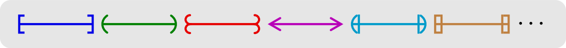

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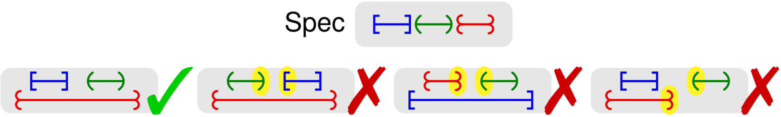

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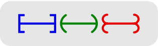
Spec 

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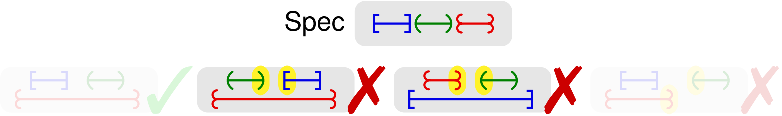

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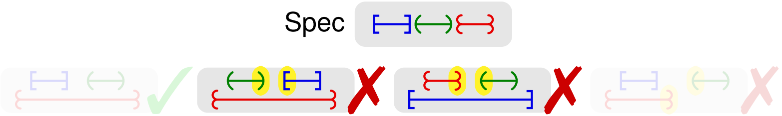

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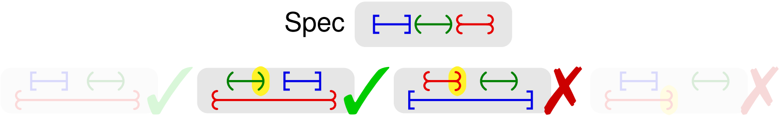

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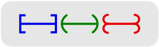
Spec 

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Interesting examples

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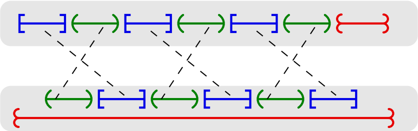

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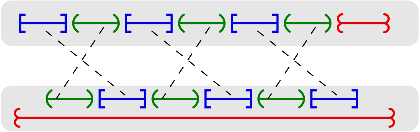

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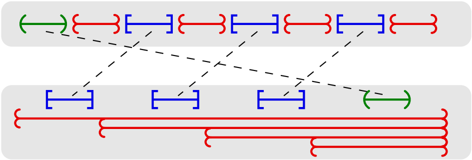

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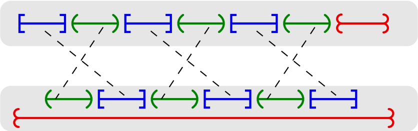

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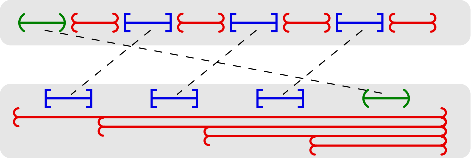

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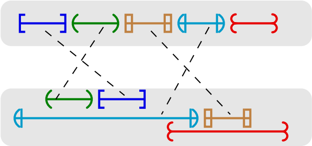
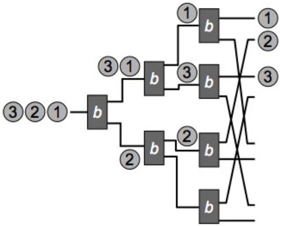
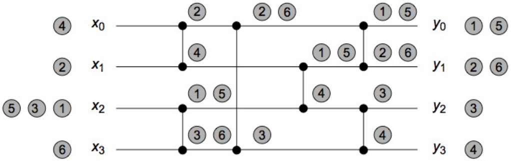


Quiescently Consistent Data Structures



- Counting networks
  - Bitonic Networks (Aspnes/Herlihy/Shavit 1991)
  - Diffracting Trees (Shavit/Zemach 1994)
  - Decrement/increment(Shavit/Touitou 1995) (Aiello/Busch/Herlihy/Mavronicolas/Shavit/Toutoui 1999)
- Stacks and Bags (aka, Pools)
  - Elimination Arrays/Trees (Shavit/Touitou 1995)
- “Almost” Linearizable
  - Experimental results
  - Theory involving max/min times (Lynch/Shavit/Shvartsman/Touitou 1996)
- *The Art of Multiprocessor Programming* (Herlihy/Shavit 2008)



N-counter (simplified from Aspnes/Herlihy/Shavit 1991)

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class Counter<N:Int> {
   field b:[0..N-1] = 0;                     // 1 balancer
   field c:Int[]    = [0, 1, ..., N-1];      // N counters
   method getAndIncrement():Int {
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      atomic { i = b; b++; }
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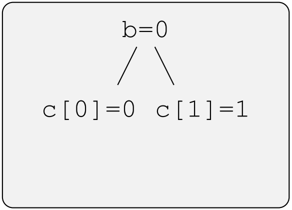

Behaves sequentially ☺

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b=1

c[0]=0 c[1]=1

$[^{\text{inc}}$

Behaves sequentially ☺

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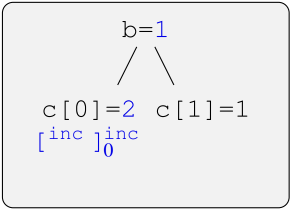

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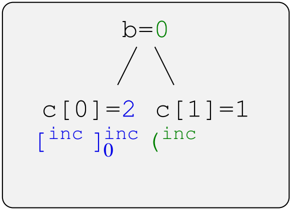

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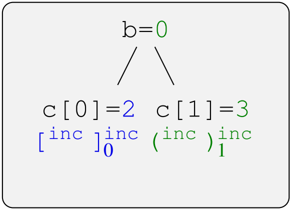

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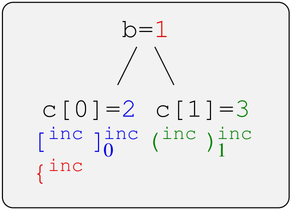

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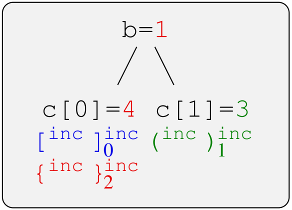

Behaves sequentially ☺

N-counter — Execution 2

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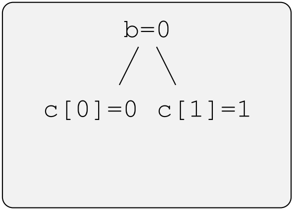

Not Linearizable ☹, but QQC ☺

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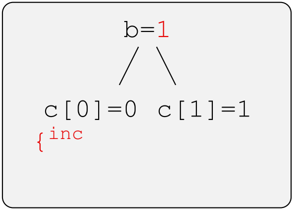

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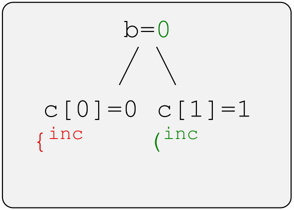

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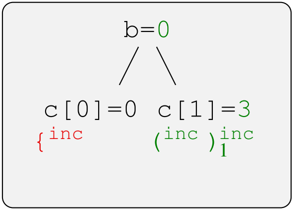

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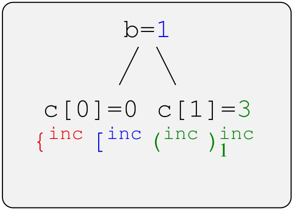

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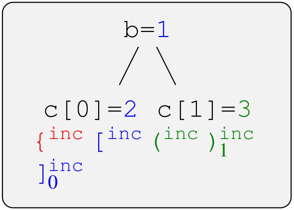

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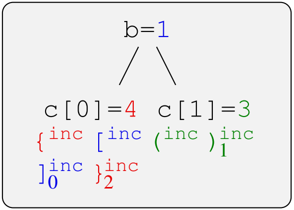

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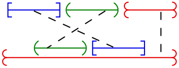

Increment/Decrement counter

```
class Counter<N:Int> {
   field b:[0..N-1] = 0;                    // 1 balancer
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   method getAndIncrement():Int {
      val i:[0..N-1];
      atomic { i = b; b++; }
      atomic { val v = c[i]; c[i] += N; return v; } } }
   method decrementAndGet():Int {
      val i:[0..N-1];
      atomic { i = b-1; b--; }
      atomic { c[i] -= N; return c[i]; } }
```

$\langle b = 0, c = [0, 1] \rangle \xrightarrow{[^{inc}} \langle b = 1, c = [0, 1] \rangle \xrightarrow{(^{inc}} \langle b = 0, c = [0, 1] \rangle$

$\xrightarrow{\{^{dec}} \langle b = 1, c = [0, 1] \rangle \xrightarrow{\langle^{dec}} \langle b = 0, c = [0, 1] \rangle$

$\xrightarrow{\rangle^{dec}_{-2}} \langle b = 0, c = [-2, 1] \rangle \xrightarrow{]^{inc}_{-2}} \langle b = 0, c = [0, 1] \rangle$

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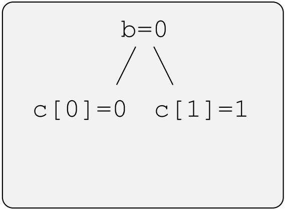

Only weak QC ☹

$dec_{-2}\, inc_{-2}\, inc_{1}\, dec_{1}$ not a permutation of any spec trace!

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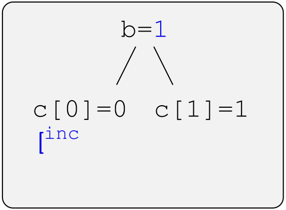

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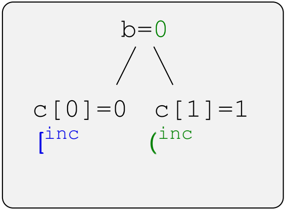

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Increment/Decrement counter

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class Counter<N:Int> {
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$$\langle b = 0, c = [0, 1]\rangle \xrightarrow{[^{\mathtt{inc}}} \langle b = 1, c = [0, 1]\rangle \xrightarrow{(^{\mathtt{inc}}} \langle b = 0, c = [0, 1]\rangle$$

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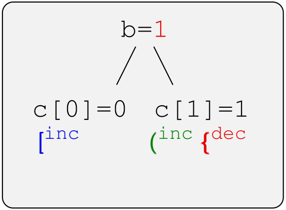
$$\xrightarrow{)^{\mathtt{dec}}_{-2}} \langle b = 0, c = [-2, 1]\rangle \xrightarrow{]^{\mathtt{inc}}_{-2}} \langle b = 0, c = [0, 1]\rangle$$

$$\xrightarrow{)^{\mathtt{inc}}_{1}} \langle b = 0, c = [0, 3]\rangle \xrightarrow{\}^{\mathtt{dec}}_{1}} \langle b = 0, c = [0, 1]\rangle$$



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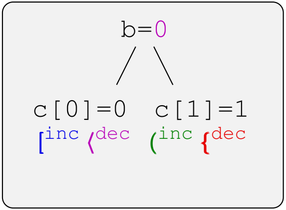

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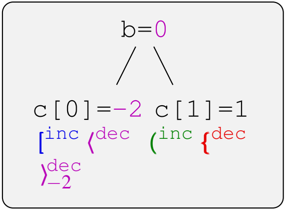

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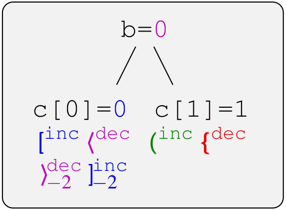

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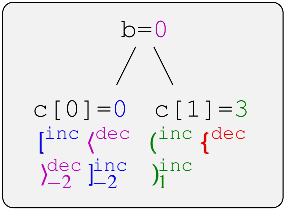

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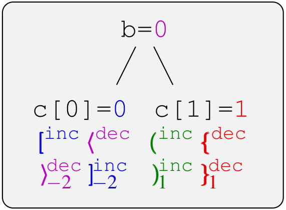

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Stack

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$$\langle \mathrm{b}=0, \mathrm{s}=[[\,],[\,]]\,\rangle \xrightarrow{[^{\mathrm{psh}}_{a}} \langle \mathrm{b}=1, \mathrm{s}=[[\,],[\,]]\,\rangle \xrightarrow{(^{\mathrm{psh}}_{b}} \langle \mathrm{b}=0, \mathrm{s}=[[\,],[\,]]\,\rangle$$

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$$\xrightarrow{>^{\mathrm{pop}}_{\mathrm{fail}}} \langle \mathrm{b}=0, \mathrm{s}=[[\,],[\,]]\,\rangle \xrightarrow{]^{\mathrm{psh}}} \langle \mathrm{b}=0, \mathrm{s}=[[a],[\,]]\,\rangle$$

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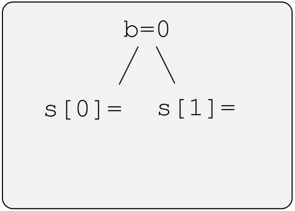

Linearizable ☺

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b=1

s[0]= $[_a^{\mathsf{psh}}$ s[1]=

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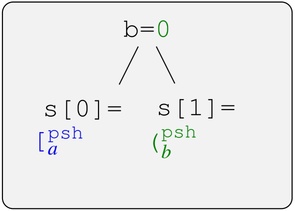

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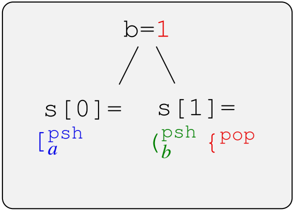

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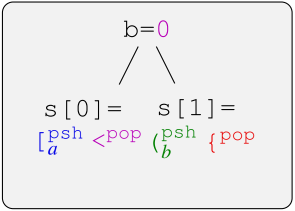

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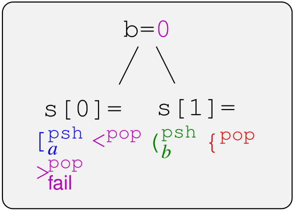

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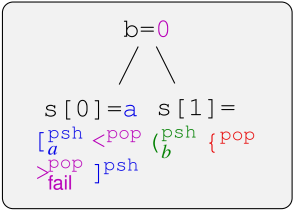

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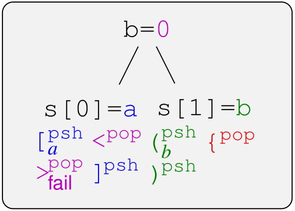

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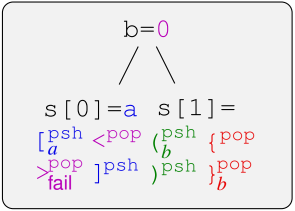
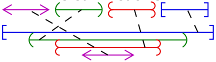


Linearizable ☺



Stack — Execution 2

```
class Stack<N:Int> {
   field b:[0..N-1] = 0;                    // 1 balancer
   field s:Stack[]  = [[], [], ..., []];    // N stacks of values
   method push(x:Object):Unit {
      val i:[0..N-1];
      atomic { i = b; b++; }
      atomic { val v = s[i].push(x); return v; } }
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      val i:[0..N-1];
      atomic { i = b-1; b--; }
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```

$$\langle \mathrm{b}=0, \mathrm{s}=[[\,],[\,]] \rangle \xrightarrow{[^{psh}_{a}} \langle \mathrm{b}=1, \mathrm{s}=[[\,],[\,]] \rangle \xrightarrow{]^{psh}} \langle \mathrm{b}=1, \mathrm{s}=[[a],[\,]] \rangle$$

$$\xrightarrow{\{^{psh}_{b}} \langle \mathrm{b}=0, \mathrm{s}=[[a],[\,]] \rangle \xrightarrow{\}^{psh}} \langle \mathrm{b}=0, \mathrm{s}=[[a],[b]] \rangle$$

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$$\xrightarrow{>^{pop}_{a}} \langle \mathrm{b}=0, \mathrm{s}=[[\_],[b]] \rangle \xrightarrow{)^{psh}} \langle \mathrm{b}=0, \mathrm{s}=[[c],[b]] \rangle$$

```
        b=0
        / \
s[0]=      s[1]=
```

Not even quiescent consistent ☹

$\longleftrightarrow$ should pop from $\longleftrightarrow$ or $\longleftrightarrow$, but not $\longleftrightarrow$

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$\langle b=0, s=[[\,],[\,]]\ \rangle \xrightarrow{[^{psh}_{a}} \langle b=1, s=[[\,],[\,]]\ \rangle \xrightarrow{]^{psh}} \langle b=1, s=[[a],[\,]]\ \rangle$

$\xrightarrow{\{^{psh}_{b}} \langle b=0, s=[[a],[\,]]\ \rangle \xrightarrow{\}^{psh}} \langle b=0, s=[[a],[b]] \rangle$

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Not even quiescent consistent ☹

↔ should pop from {—} or (—), but not [—]

```
      b=1
      / \
 s[0]=   s[1]=
[psh
 a
```

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```
        b=1
        / \
s[0]=a   s[1]=
```
$[^{\mathtt{psh}}_{a}\ ]^{\mathtt{psh}}$

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$$\xrightarrow{\{^{psh}_{b}} \langle b = 0, s = [[a],[\,]]\rangle \xrightarrow{\}^{psh}} \langle b = 0, s = [[a],[b]]\rangle$$

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$$\xrightarrow{>^{pop}_{a}} \langle b = 0, s = [[\_],[b]]\rangle \xrightarrow{)^{psh}} \langle b = 0, s = [[c],[b]]\rangle$$

b=0
/ \
s[0]=a s[1]=
$[^{psh}_{a}$ $]^{psh}$ $\{^{psh}_{b}$

Not even quiescent consistent ☹

⟷ should pop from ⟷ or ⟷, but not ⟷

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b=0

/ \

s[0]=a s[1]=b

$[^{psh}_{a}\ ]^{psh}$ $\{^{psh}_{b}\ \}^{psh}$

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```

$$\langle b = 0, s = [[\,], [\,]]\ \rangle \xrightarrow{[_a^{psh}} \langle b = 1, s = [[\,], [\,]]\ \rangle \xrightarrow{]^{psh}} \langle b = 1, s = [[a], [\,]]\ \rangle$$

$$\xrightarrow{\{_b^{psh}} \langle b = 0, s = [[a], [\,]]\ \rangle \xrightarrow{\}^{psh}} \langle b = 0, s = [[a], [b]] \rangle$$

$$\xrightarrow{(_c^{psh}} \langle b = 1, s = [[a], [b]] \rangle \xrightarrow{<^{pop}} \langle b = 0, s = [[a], [b]] \rangle$$

$$\xrightarrow{>_a^{pop}} \langle b = 0, s = [[\_], [b]] \rangle \xrightarrow{)^{psh}} \langle b = 0, s = [[c], [b]] \rangle$$

$b=1$

/ \

$s[0]=a$ $s[1]=b$

$[_a^{psh}$ $]^{psh}$ $\{_b^{psh}$ $\}^{psh}$

$)^{psh}$

Not even quiescent consistent ☹

$\longleftrightarrow$ should pop from $\{\!\!\longrightarrow\!\!\}$ or $(\!\longleftrightarrow\!)$, but not $[\!\longrightarrow\!]$

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Not even quiescent consistent ☹

↔ should pop from {—} or (—), but not [—]

b=0

/ \

s[0]= a s[1]= b

$[^{psh}_{a}$ $]^{psh}$ $\{^{psh}_{b}$ $\}^{psh}$

$)^{psh}$ $<^{pop}$

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$\longleftrightarrow$ should pop from $\{\!\!\longleftrightarrow\!\!\}$ or $(\longleftrightarrow)$, but not $[\longleftrightarrow]$

b=0

/ \

s[0]= s[1]= b

$[^{\mathrm{psh}}_{a}\ ]^{\mathrm{psh}}$ $\{^{\mathrm{psh}}_{b}\ \}^{\mathrm{psh}}$

$)^{\mathrm{psh}}\ <^{\mathrm{pop}}$

$>^{\mathrm{pop}}_{a}$

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$$\xrightarrow{>_a^{\mathtt{pop}}} \langle \mathrm{b}=0, \mathrm{s}=[[\_],[b]]\rangle \xrightarrow{)^{\mathtt{psh}}} \langle \mathrm{b}=0, \mathrm{s}=[[c],[b]]\rangle$$

b=0

/ \

s[0]= c s[1]= b

$[_a^{\mathtt{psh}}$ $]^{\mathtt{psh}}$ $\{_b^{\mathtt{psh}}$ $\}^{\mathtt{psh}}$

$)^{\mathtt{psh}}$ $<^{\mathtt{pop}}$

$>_a^{\mathtt{pop}}$ $)^{\mathtt{psh}}$

Not even quiescent consistent ☹

$\leftrightarrow$ should pop from $\{\!-\!\}$ or $(\!-\!)$, but not $[\!-\!]$

Results

- Three characterizations of QQC
  - Call-to-return (given earlier)
  - Return-to-call (à la Herlihy/Wing)
  - Proxy for sequential implementation (flat combiner + speculation)
    - Single thread accesses sequential structure
    - Upon receiving actual call, speculatively execute any method with any args
    - Only return when speculative call matches actual call
- Proof of compositionality
  - Global constraints that are solvable because of “flow” properties
- Proofs and counterexamples for tree-based structures
  - Increment/decrement N-counter (weak QC)
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  - General N-stack (QC)
  - “Properly popped” N-stack (QQC)
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Return-to-call characterization

Linearizability:
$\forall prefix/suffix = exec$

$\forall ret \in prefix$
$\forall call \in suffix$

$$ret \xrightarrow{exec} call \quad \text{implies} \quad ret \xrightarrow{spec} call$$

Return-to-call characterization

Quiescent consistency:

$\forall prefix/suffix = exec$

if $prefix$ has 0 open calls, then

$\forall ret \in prefix$

$\forall call \in suffix$

$$ret \xrightarrow{exec} call \quad \text{implies} \quad ret \xrightarrow{spec} call$$

Return-to-call characterization

QQC:
$\forall prefix/suffix = exec$
if $prefix$ has k open/early calls, then there exists $|ignoredCalls| \leq k$
$\forall ret \in prefix$
$\forall call \in suffix - ignoredCalls$

$$ret \xrightarrow{exec} call \quad \text{implies} \quad ret \xrightarrow{spec} call$$

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Related work

- *Quantitative Relaxation of Concurrent Data Structures*
  (Henzinger/Kirsch/Payer/Sezgin/Sokolova 2013)
- Incomparable (Examples from Sezgin)
  - Stack that is 1-out-of-order but not QQC:

    $$(_c^{\mathrm{psh}}\ [_a^{\mathrm{psh}}\ ]^{\mathrm{psh}}\ \{_b^{\mathrm{psh}}\ \}^{\mathrm{psh}}\ )^{\mathrm{psh}}\ <^{\mathrm{pop}}\ >_a^{\mathrm{pop}}$$

    However,

    $$(_c^{\mathrm{psh}}\ [_a^{\mathrm{psh}}\ ]^{\mathrm{psh}}\ \{_b^{\mathrm{psh}}\ \}^{\mathrm{psh}}\ <^{\mathrm{pop}}\ >_a^{\mathrm{pop}}\ )^{\mathrm{psh}}$$

    is QQC w.r.t. the stack spec

    $$\{_b^{\mathrm{psh}}\ \}^{\mathrm{psh}}\ [_a^{\mathrm{psh}}\ ]^{\mathrm{psh}}\ <^{\mathrm{pop}}\ >_a^{\mathrm{pop}}\ (_c^{\mathrm{psh}}\ )^{\mathrm{psh}}$$

  - For stacks, it may be that QQC is finer that n-out-of-order (arbitrary n)
  - Queue that is QQC but not $(n-1)$-out-of-order:

    $$(_a^{\mathrm{psh}}\ [_{b_1}^{\mathrm{psh}}\ ]^{\mathrm{psh}}\ [_{b_1}^{\mathrm{psh}}\ ]^{\mathrm{psh}}\ \cdots [_{b_n}^{\mathrm{psh}}\ ]^{\mathrm{psh}}\ \{_c^{\mathrm{psh}}\ \}^{\mathrm{psh}}\ <^{\mathrm{pop}}\ >_c^{\mathrm{pop}}\ )^{\mathrm{psh}}$$

    This is QQC w.r.t.

    $$\{_c^{\mathrm{psh}}\ \}^{\mathrm{psh}}\ [_{b_1}^{\mathrm{psh}}\ ]^{\mathrm{psh}}\ [_{b_1}^{\mathrm{psh}}\ ]^{\mathrm{psh}}\ \cdots [_{b_n}^{\mathrm{psh}}\ ]^{\mathrm{psh}}\ <^{\mathrm{pop}}\ >_c^{\mathrm{pop}}\ (_a^{\mathrm{psh}}\ )^{\mathrm{psh}}$$

Related work

- *Quantitative Relaxation of Concurrent Data Structures*
  (Henzinger/Kirsch/Payer/Sezgin/Sokolova 2013)
- Incomparable (Examples from Sezgin)
  - Stack that is 1-out-of-order but not QQC:

    $$(_c^{\mathrm{psh}}\ [_a^{\mathrm{psh}}\ ]^{\mathrm{psh}}\ \{_b^{\mathrm{psh}}\ \}^{\mathrm{psh}}\ )^{\mathrm{psh}}\ <^{\mathrm{pop}}\ >_a^{\mathrm{pop}}$$

    However,

    $$(_c^{\mathrm{psh}}\ [_a^{\mathrm{psh}}\ ]^{\mathrm{psh}}\ \{_b^{\mathrm{psh}}\ \}^{\mathrm{psh}}\ <^{\mathrm{pop}}\ >_a^{\mathrm{pop}}\ )^{\mathrm{psh}}$$

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    $$\{_b^{\mathrm{psh}}\ \}^{\mathrm{psh}}\ [_a^{\mathrm{psh}}\ ]^{\mathrm{psh}}\ <^{\mathrm{pop}}\ >_a^{\mathrm{pop}}\ (_c^{\mathrm{psh}}\ )^{\mathrm{psh}}$$

  - For stacks, it may be that QQC is finer that n-out-of-order (arbitrary n)
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Proxy characterization code

```
interface Object {
   method run(i:Invocation):Response;
   method predict():Invocation;   }
class QQCProxy<o:Object> {
   field called:ThreadSafeMultiMap<Invocation,Semaphore> = [];
   field returned:ThreadSafeMap   <Semaphore, Response>  = [];
   method run(i:Invocation):Response { // proxy for external access to o
     val m:Semaphore = [];
     called.add(i, m);
     m.wait();
     return returned.remove(m); }
   thread { // single thread to interact with o
     val received:MultiMap<Invocation,Semaphore> = [];
     val executed:MultiMap<Invocation,Response>  = [];
     repeatedly choose {
        choice if called.notEmpty() {
           received.add(called.removeAny());
           val i:Invocation = o.predict();
           val r:Response   = o.run(i);
           executed.add(i, r); }
        choice if exists i in received.keys() intersect executed.keys() {
           val m:Semaphore = received.remove(i);
           val r:Response  = executed.remove(i);
           returned.add(m, r);
           m.signal(); } } } }
```